Introduction to the ZX-calculus

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Inria & Loria, Nancy

QCOMICAL school, 3rd November 2025

Outline

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Motivation: quantum circuit optimisation

Quantum computational resources are limited, so we need to use them efficiently.

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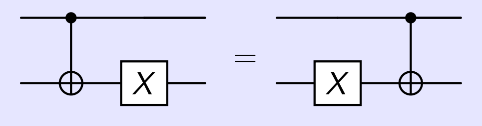
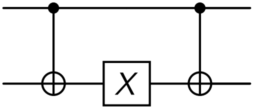
Quantum computational resources are limited, so we need to use them efficiently.

- Given a quantum circuit, can we find a more efficient circuit that describes the same linear map?
- If someone gives you a circuit and claims it is a more efficient version of the circuit you want to run, how can you check that the two circuits really describe the same linear map?

Rewriting quantum circuits





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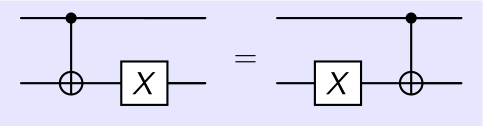

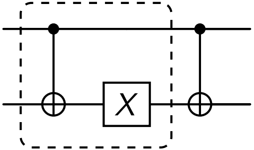

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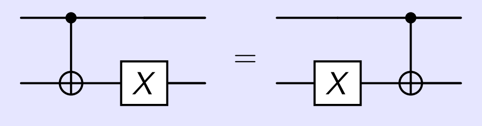

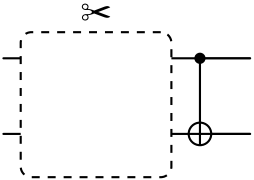

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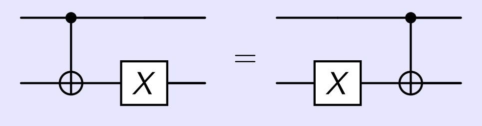

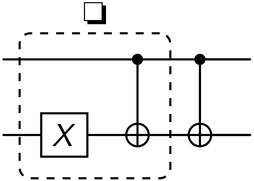

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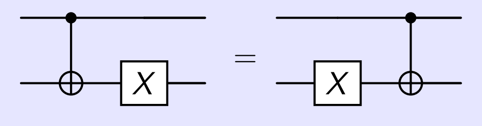

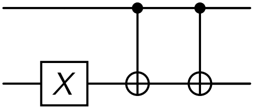

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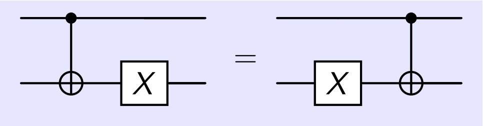

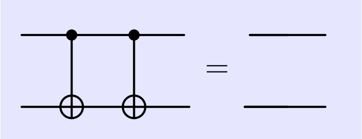

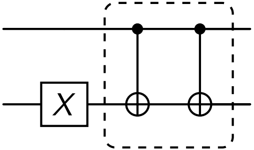

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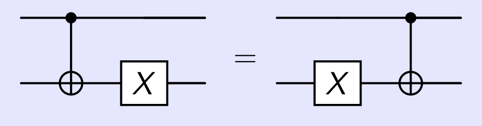

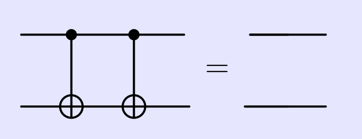

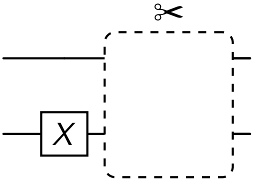

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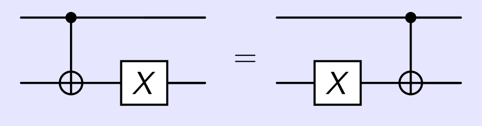

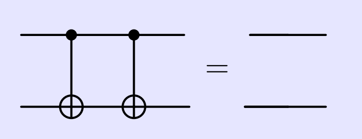

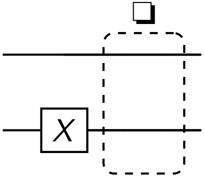

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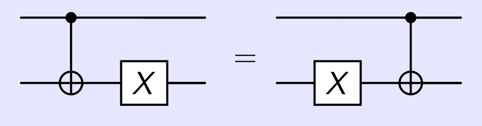
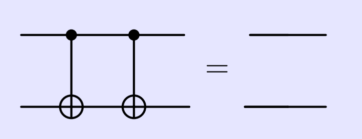
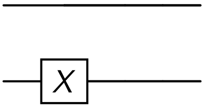
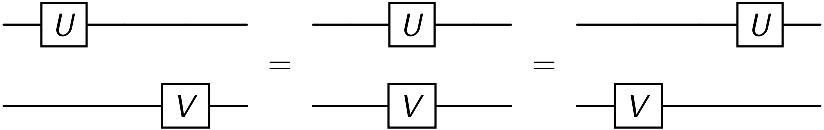






Some circuit equations are particularly intuitive



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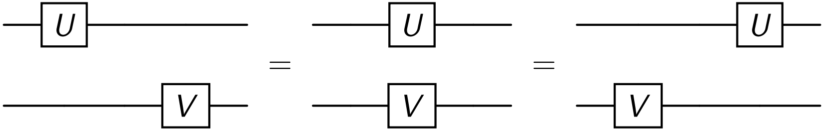
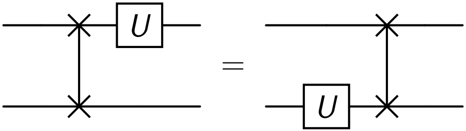


Notation can make a difference, for example the two symbols for the SWAP gate



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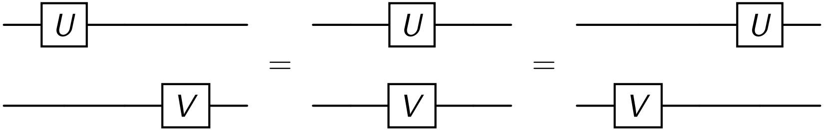

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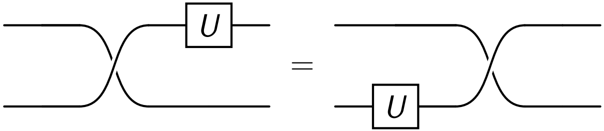
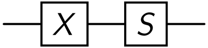
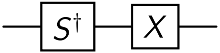


Equality up to scalar factor



$$\rightsquigarrow \quad \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix}$$



$$\rightsquigarrow \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & -i \\ 1 & 0 \end{pmatrix} = -i \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix}$$

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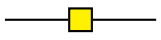
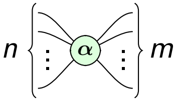
The ZX-calculus components: (mostly) spiders instead of gates

Hadamard gate

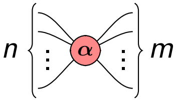
[Coecke & Duncan 2008]



Z-spider



X-spider



The ZX-calculus components: (mostly) spiders instead of gates

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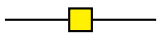
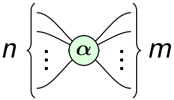
 $\rightsquigarrow$ $|+\rangle\langle 0| + |-\rangle\langle 1| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Z-spider



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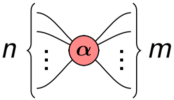

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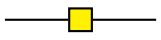

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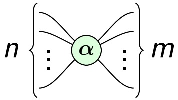

$$\rightsquigarrow \quad |\underbrace{0\ldots0}_{m}\rangle\langle\underbrace{0\ldots0}_{n}| + e^{i\alpha}|\underbrace{1\ldots1}_{m}\rangle\langle\underbrace{1\ldots1}_{n}| = \begin{pmatrix} 1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & e^{i\alpha} \end{pmatrix}$$

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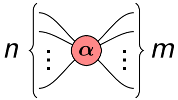

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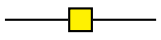

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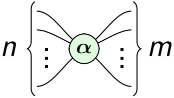

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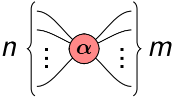
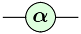


$$\rightsquigarrow \quad |\underbrace{+\ldots+}_{m}\rangle\langle\underbrace{+\ldots+}_{n}| + e^{i\alpha}|\underbrace{-\ldots-}_{m}\rangle\langle\underbrace{-\ldots-}_{n}|$$

A closer look at Z-spiders: unitaries



$$\rightsquigarrow \quad |0\rangle\langle 0| + e^{i\alpha}|1\rangle\langle 1| \quad = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

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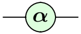
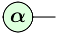


$$\rightsquigarrow \quad |0\rangle\langle 0| + e^{i\alpha}\,|1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

$$\rightsquigarrow \quad |0\rangle\langle 0| - |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$$

A closer look at Z-spiders: states



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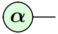

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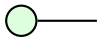

$$\rightsquigarrow \quad |0\rangle + |1\rangle = \sqrt{2}\,|+\rangle \qquad = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

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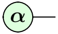

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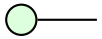

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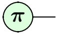
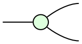


$$\rightsquigarrow \quad |0\rangle - |1\rangle = \sqrt{2}\,|-\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

A closer look at Z-spiders: the copy map



$$\rightsquigarrow \quad |00\rangle\langle 0| + |11\rangle\langle 1| \quad = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

A closer look at X-spiders: unitaries

$$\rightsquigarrow \qquad |+\rangle\langle+| + e^{i\alpha}\,|-\rangle\langle-| \qquad = e^{i\alpha/2}\begin{pmatrix} \cos\frac{\alpha}{2} & i\sin\frac{\alpha}{2} \\ i\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}$$

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A closer look at X-spiders: states

$\rightsquigarrow \quad |+\rangle + |-\rangle = \sqrt{2}\,|0\rangle \qquad = \sqrt{2}\begin{pmatrix}1\\0\end{pmatrix}$

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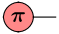
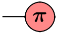


$$\rightsquigarrow \quad |+\rangle - |-\rangle = \sqrt{2}\,|1\rangle \qquad = \sqrt{2}\begin{pmatrix}0\\1\end{pmatrix}$$

A closer look at X-spiders: measurement outcomes

$$\rightsquigarrow \quad \langle +| + \langle -| = \sqrt{2}\,\langle 0| \quad = \sqrt{2}\begin{pmatrix}1 & 0\end{pmatrix}$$



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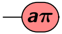
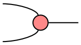


$$\rightsquigarrow \quad \langle +| + (-1)^a\,\langle -| \quad = \sqrt{2}\begin{pmatrix}1-a & a\end{pmatrix}$$

A closer look at X-spiders: the parity map



$$\rightsquigarrow \quad |+\rangle\langle ++| + |-\rangle\langle --| \quad = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Wires in the ZX-calculus

 $\rightsquigarrow \quad |0\rangle\langle 0| + |1\rangle\langle 1| \qquad = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

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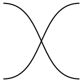
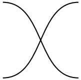


$$\rightsquigarrow \quad |00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10| + |11\rangle\langle 11| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Wires in the ZX-calculus

$$\rightsquigarrow \quad |0\rangle\langle 0| + |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

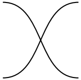


$$\rightsquigarrow \quad |00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10| + |11\rangle\langle 11| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\rightsquigarrow \quad \langle 00| + \langle 11| = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$$

Wires in the ZX-calculus

$$\rightsquigarrow \quad |0\rangle\langle 0| + |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

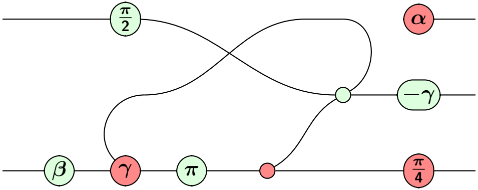


$$\rightsquigarrow \quad |00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10| + |11\rangle\langle 11| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

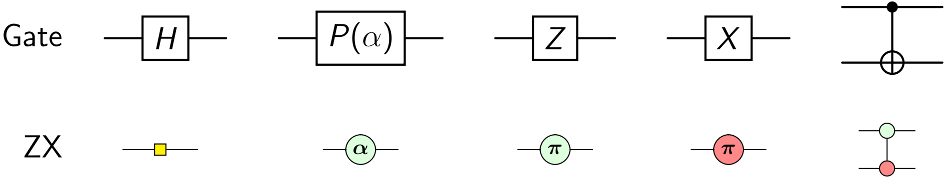
$$\rightsquigarrow \quad \langle 00| + \langle 11| = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\rightsquigarrow \quad |00\rangle + |11\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

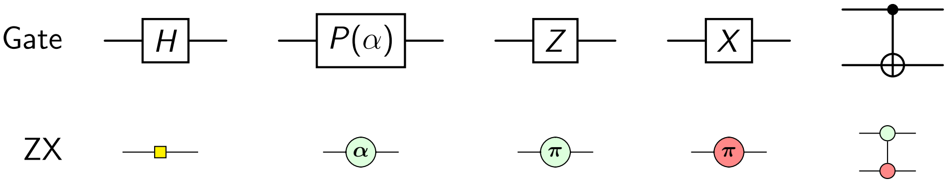
Building ZX-diagrams

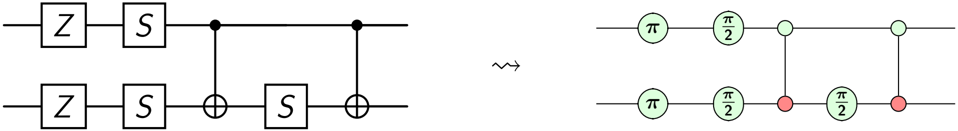


Translating circuits into ZX-diagrams



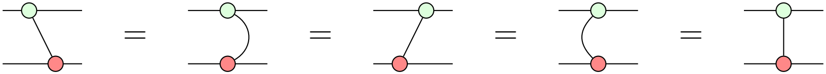
Translating circuits into ZX-diagrams



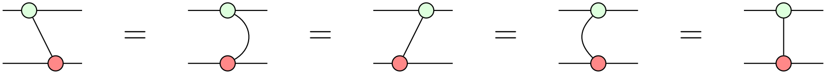


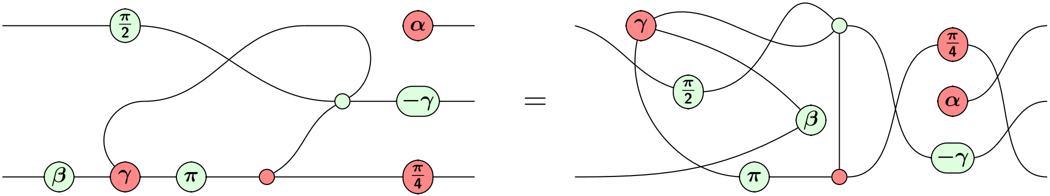
Outline

Only connectivity matters

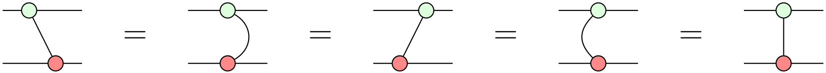


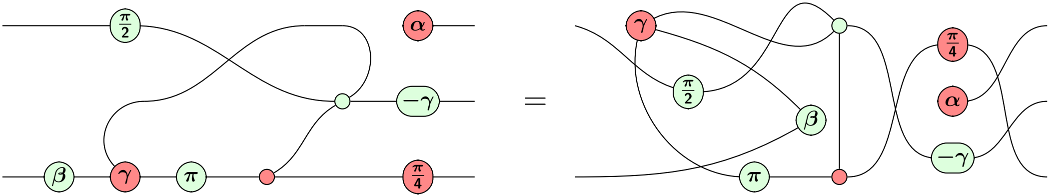
Only connectivity matters





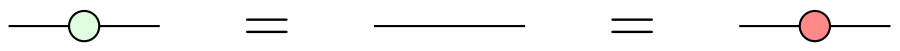
Only connectivity matters



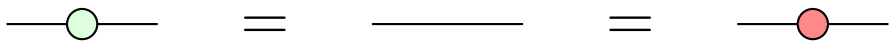


This is made mathematically rigorous using monoidal category theory.

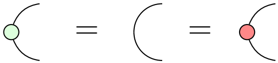
The identity rule



The identity rule

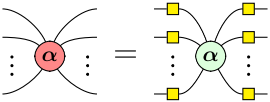


Because only connectivity matters, we also get:

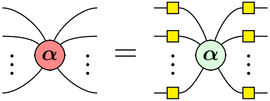


and

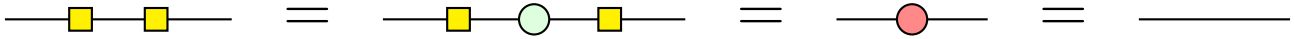
The colour change rule



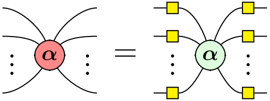
The colour change rule



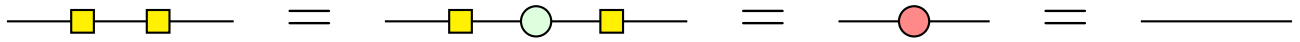
Two Hadamard gates in a row cancel:



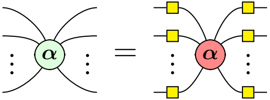
The colour change rule



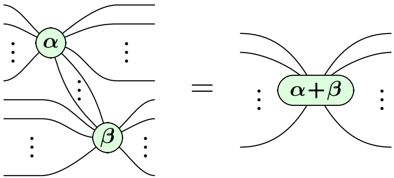
Two Hadamard gates in a row cancel:



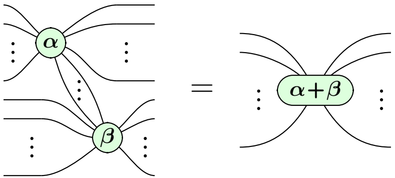
All diagram equations hold with Z- and X-spiders swapped, for example:



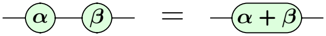
The spider rule



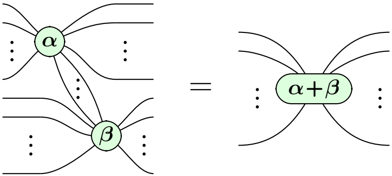
The spider rule



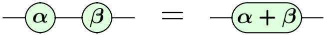
For example:

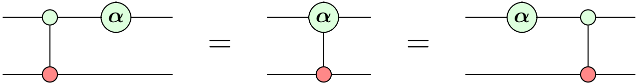


The spider rule

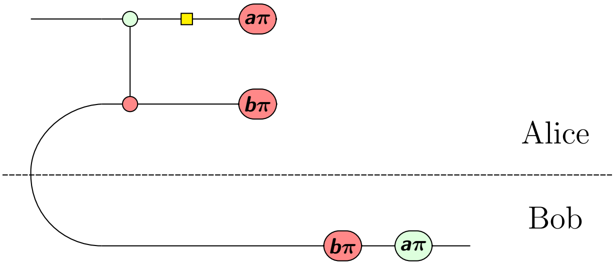


For example:

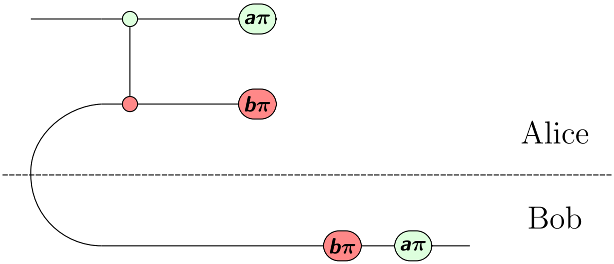




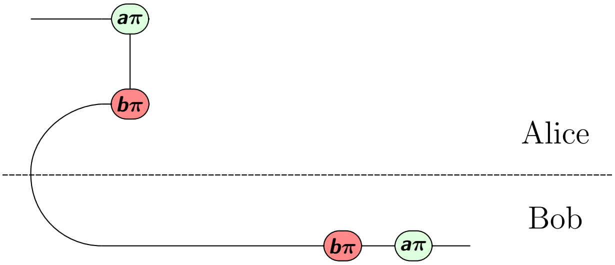
Example: quantum teleportation



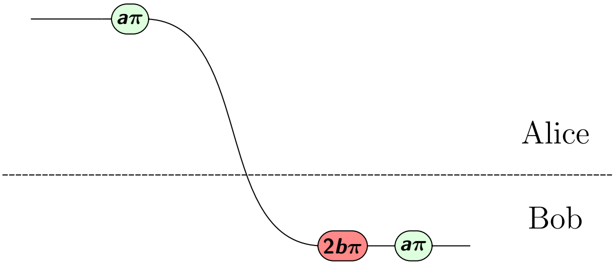
Example: quantum teleportation



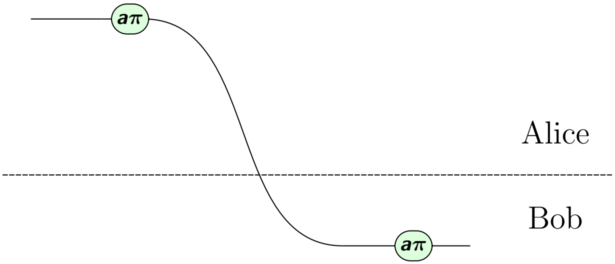
Example: quantum teleportation



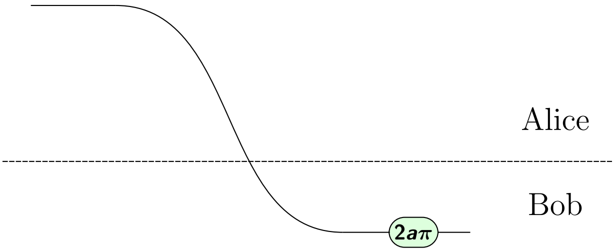
Example: quantum teleportation



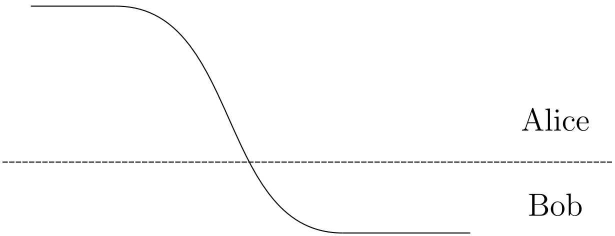
Example: quantum teleportation



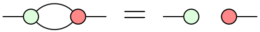
Example: quantum teleportation



Example: quantum teleportation



The Hopf rule



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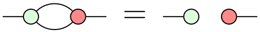
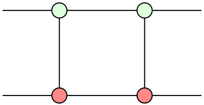


Example: two CNOTs cancel



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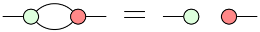

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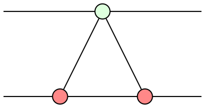

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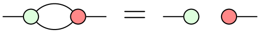

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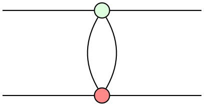

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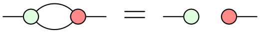

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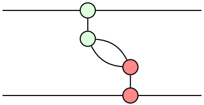
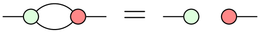


The Hopf rule



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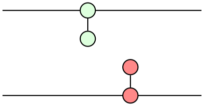
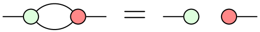


The Hopf rule



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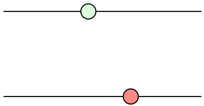
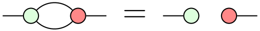


The Hopf rule



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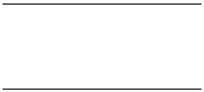
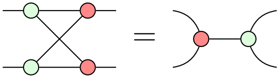


The bialgebra rule



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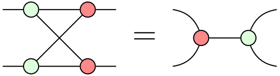
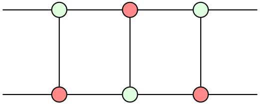
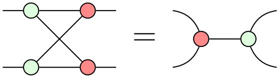


Example: three CNOTs make a SWAP



The bialgebra rule



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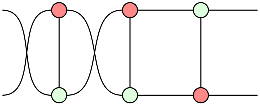
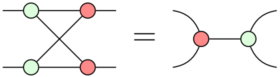


The bialgebra rule



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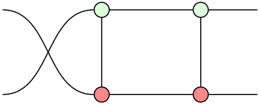

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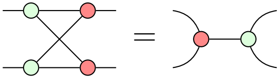

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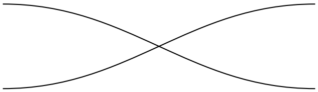
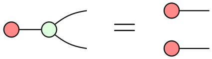


The copy rule



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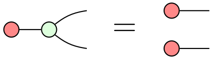
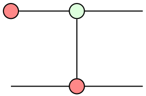


Example: plugging $|0\rangle$ into the control of a CNOT means the gate does nothing



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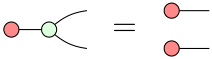

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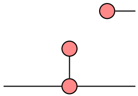

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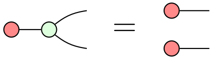

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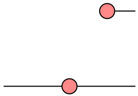

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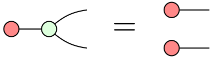

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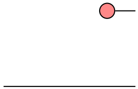
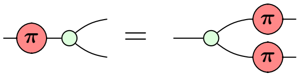


The π-copy rule



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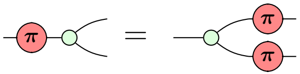
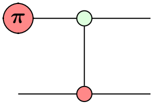
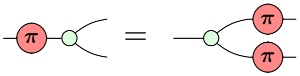


Example: plugging $|1\rangle$ into the control of a CNOT flips the second qubit



The π-copy rule



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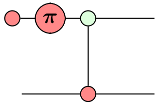

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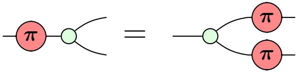

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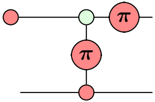

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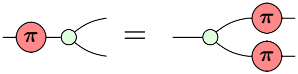

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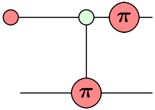

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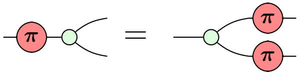

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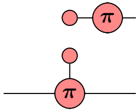

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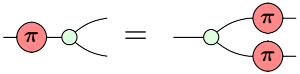

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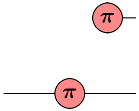
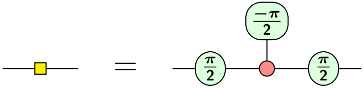


Two more rules about single-qubit operations



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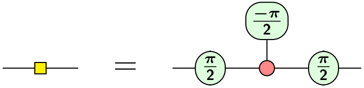
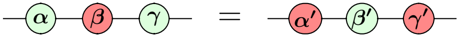
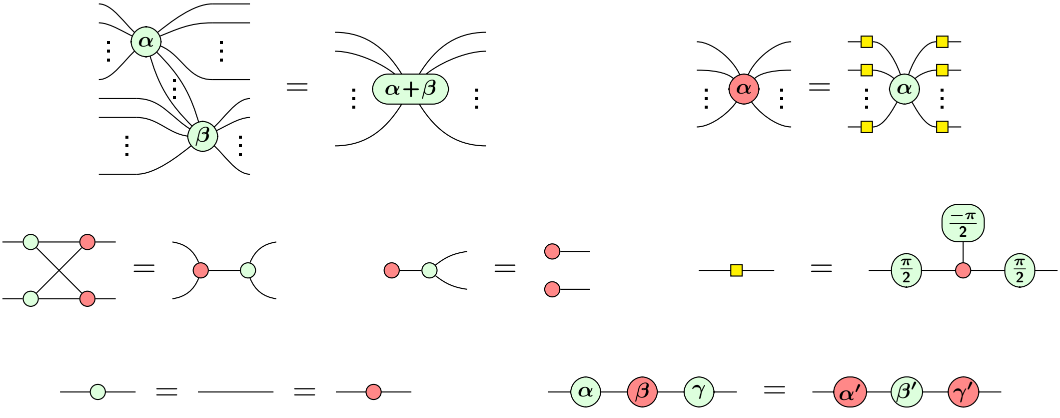


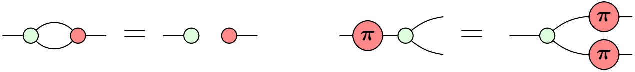


(with a complicated relationship between the phase angles)

The complete set of (scalar-free) ZX-calculus rewrite rules



Only connectivity matters.



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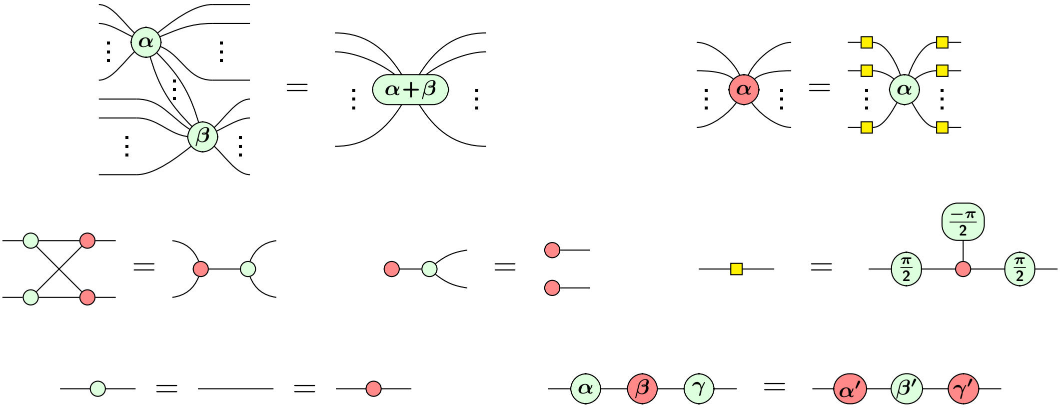

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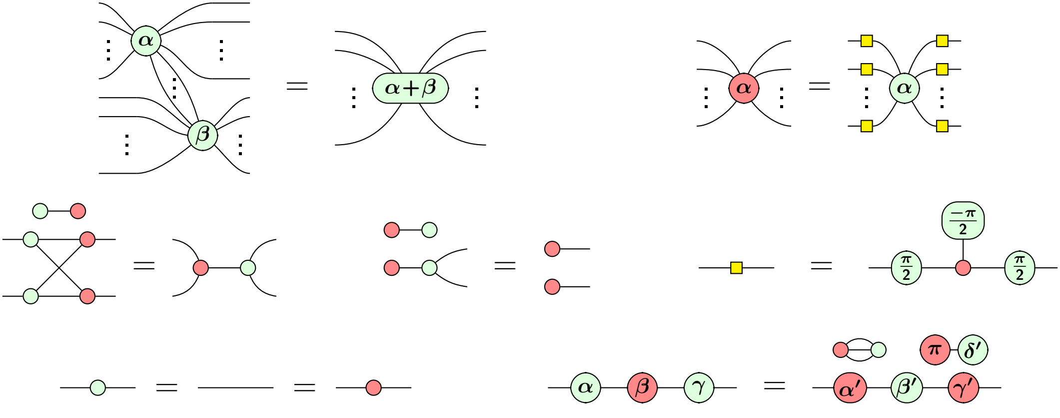

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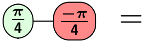


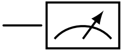
[B. 2014; Ng and Wang 2017; Jeandel, Perdrix, Vilmart 2018; Vilmart 2019]

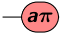
Outline

Outline

Single-qubit measurements

Have already seen computational basis measurements:





$$\rightsquigarrow \quad \sqrt{2}\begin{pmatrix} a & 1-a \end{pmatrix} = \begin{cases} \sqrt{2}\,\langle 0| & \text{if } a = 0 \\ \sqrt{2}\,\langle 1| & \text{if } a = 1 \end{cases}$$

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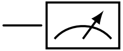

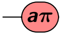

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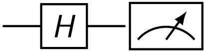

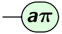

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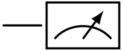

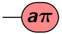

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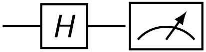

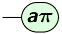

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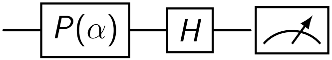

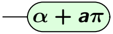

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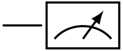

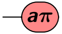

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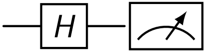

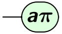

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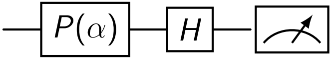

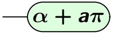

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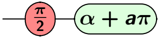
Similarly, can also consider more complicated ZX-diagrams to represent a single measurement, e.g. 

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Single-qubit measurements

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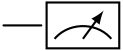

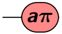

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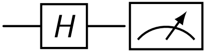

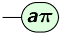

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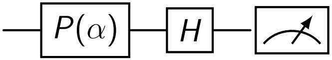

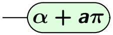

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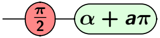
Similarly, can also consider more complicated ZX-diagrams to represent a single measurement, e.g. 

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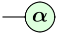
Leave out variable to represent post-selected measurements, e.g. 

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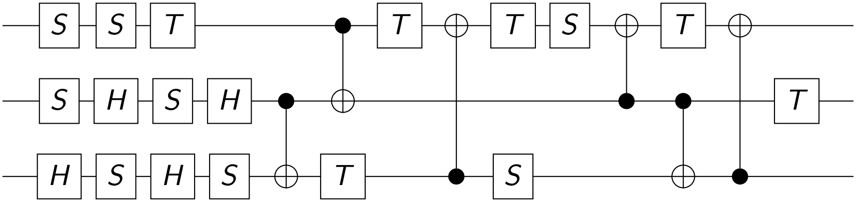
Optimising quantum computations using the ZX-calculus



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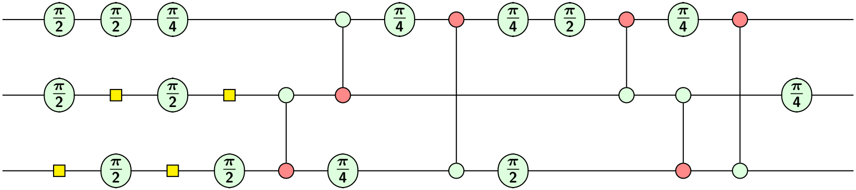

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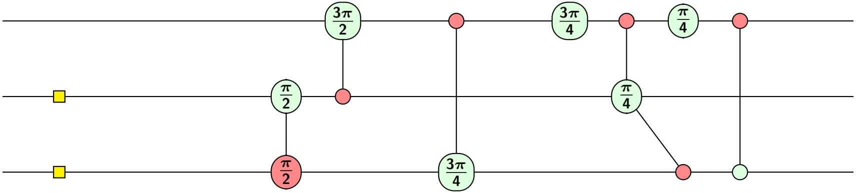

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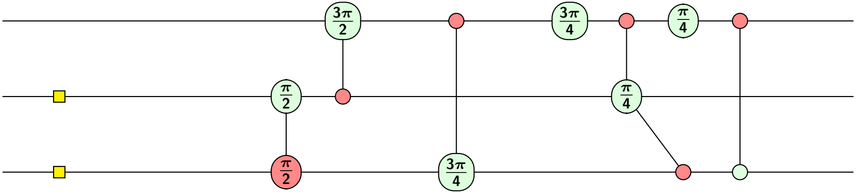

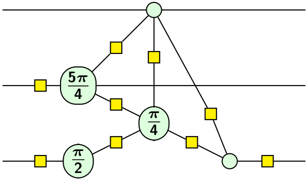

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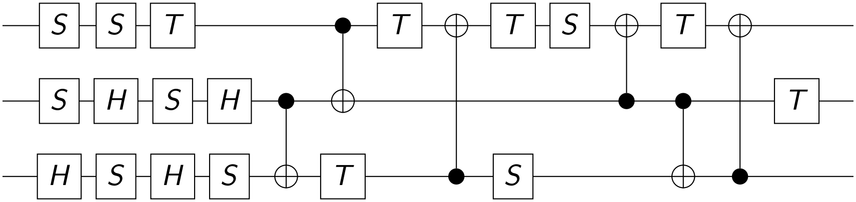
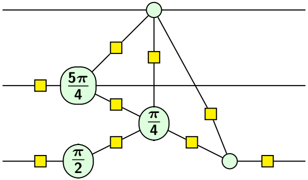




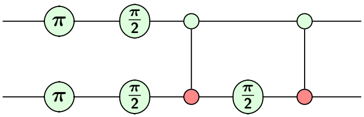
Problem: translating ZX-diagrams to circuits is #P-hard [de Beaudrap et al. 2022]

Two models of quantum computation

Two models of quantum computation

quantum circuit model

[Deutsch 1989]



Two models of quantum computation

quantum circuit model

[Deutsch 1989]

- initialise classical state
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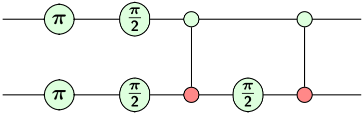

Two models of quantum computation

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- initialise classical state $0 \ldots 0$
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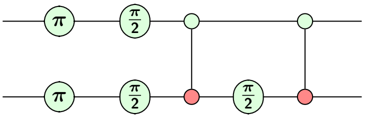

Two models of quantum computation

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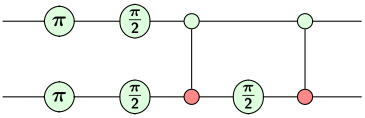
- simple measurements read out data at end



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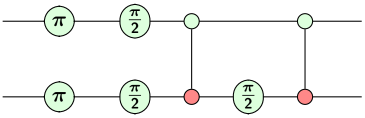
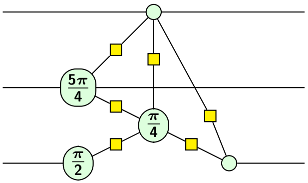


one-way model

[Raussendorf & Briegel 2001]



Two models of quantum computation

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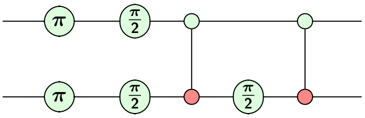
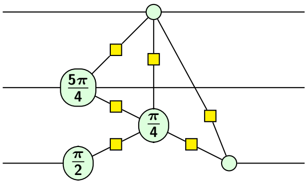


one-way model

[Raussendorf & Briegel 2001]

- initialise entangled 'graph state' (can be made independent of computation)



Two models of quantum computation

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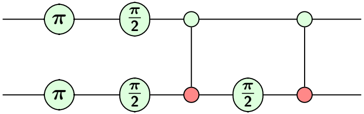

one-way model

[Raussendorf & Briegel 2001]

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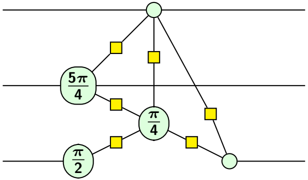

Two models of quantum computation

quantum circuit model

[Deutsch 1989]

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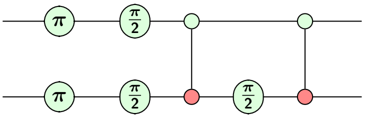

one-way model

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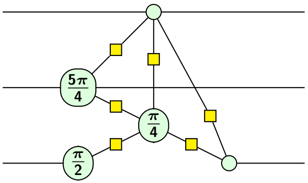
- initialise entangled 'graph state' (can be made independent of computation)
- computation driven by successive adaptive single-qubit measurements
- if goal is state preparation, may need Pauli corrections at the end



Graph states as ZX-diagrams

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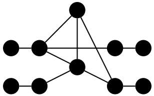

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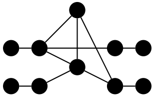
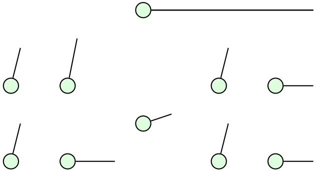


- For each vertex, prepare a qubit in the state $|+\rangle$.



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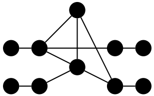

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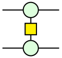
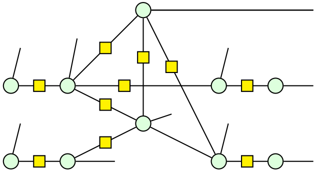
- For each edge, apply a controlled-Z gate  .



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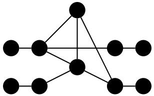

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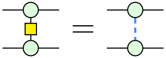
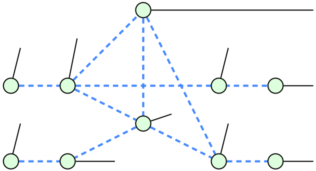
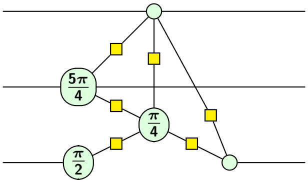
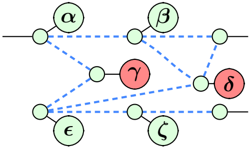
- For each edge, apply a controlled-Z gate  .



Representing one-way computations in the ZX-calculus





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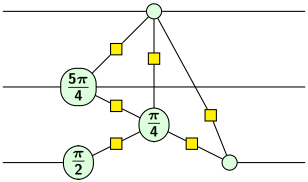

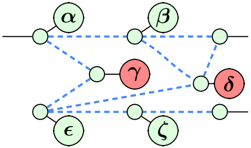

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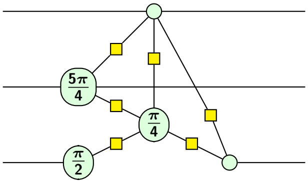

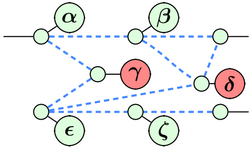

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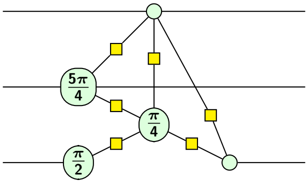

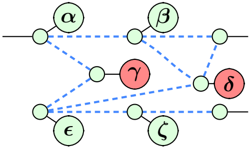

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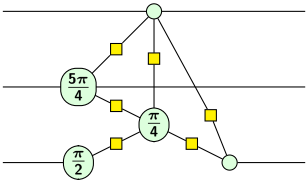

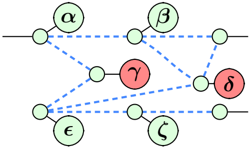

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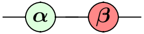
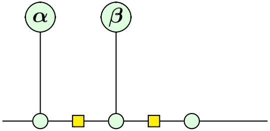
Can allow different measurement effects:

Correcting undesired measurement outcomes

Suppose we want to implement the operation  in the one-way model:



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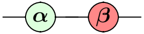
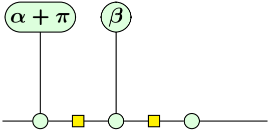
Suppose we want to implement the operation  in the one-way model:



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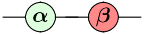
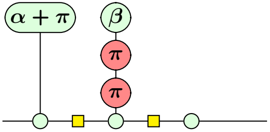
Suppose we want to implement the operation  in the one-way model:



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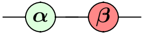
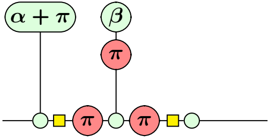
Suppose we want to implement the operation  in the one-way model:



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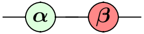
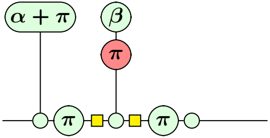
Suppose we want to implement the operation  in the one-way model:



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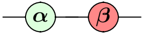
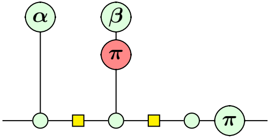
Suppose we want to implement the operation  in the one-way model:



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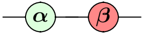
Suppose we want to implement the operation 

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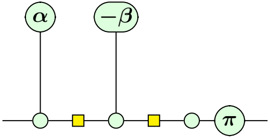

Suppose the first measurement yields the undesired outcome. This can be compensated by flipping the second angle to $-\beta$ and applying Z to the output.

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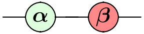
Suppose we want to implement the operation 

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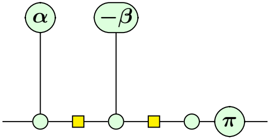

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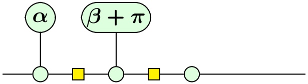

Similarly, suppose the second measurement yields the undesired outcome.

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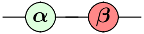
Suppose we want to implement the operation 

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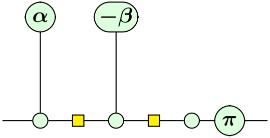

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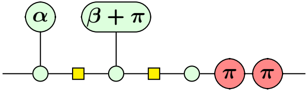

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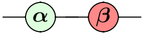
Suppose we want to implement the operation 

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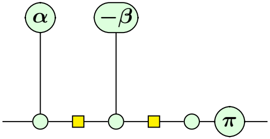

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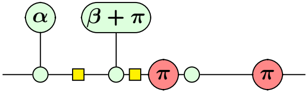

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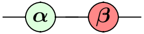
Suppose we want to implement the operation 

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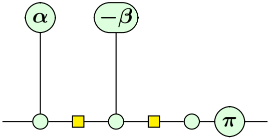

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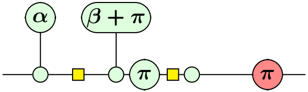

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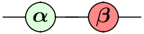
Suppose we want to implement the operation 

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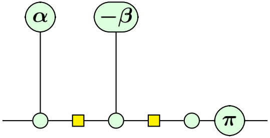

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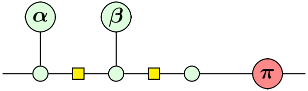

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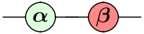
Suppose we want to implement the operation 

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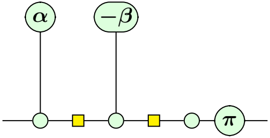

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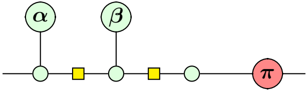

Similarly, suppose the second measurement yields the undesired outcome. This can be corrected by applying X to the output, without affecting the first measurement. The two correction procedures are compatible with each other.

Determinism in the one-way model allows circuit extraction

If for a one-way computation, there exist:

- a partial order on the qubits, and

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There exists an efficient algorithm which 'extracts a circuit' from any ZX-diagram that corresponds to a one-way computation with flow.

[Duncan et al. 2020; B. et al. 2021; Simmons 2021]

Outline

The quantum bit-flip code

Goal: want to protect quantum information against bit flip error $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

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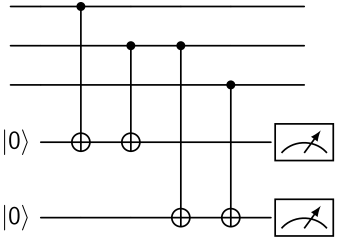

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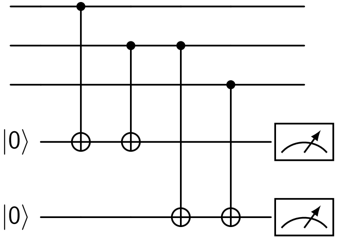
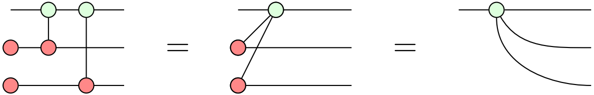
Measuring parities does not destroy the superposition:



$$\alpha\,|000\rangle + \beta\,|111\rangle \rightsquigarrow 00$$
$$\alpha\,|100\rangle + \beta\,|011\rangle \rightsquigarrow 10$$
$$\alpha\,|010\rangle + \beta\,|101\rangle \rightsquigarrow 11$$
$$\alpha\,|001\rangle + \beta\,|110\rangle \rightsquigarrow 01$$

The bit flip code in the ZX-calculus

Encoder:



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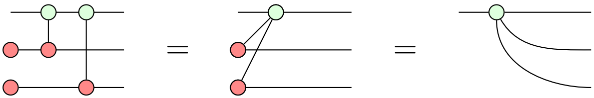
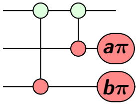


Use the adjoint of this map to decode; this will involve measurements:



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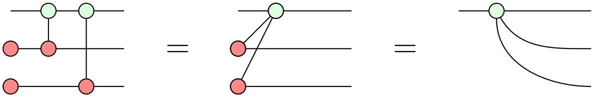

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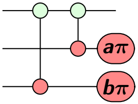
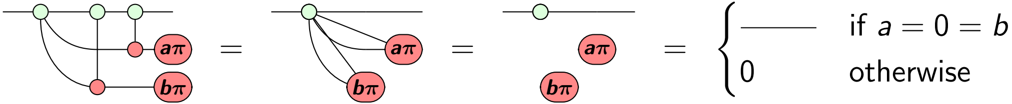
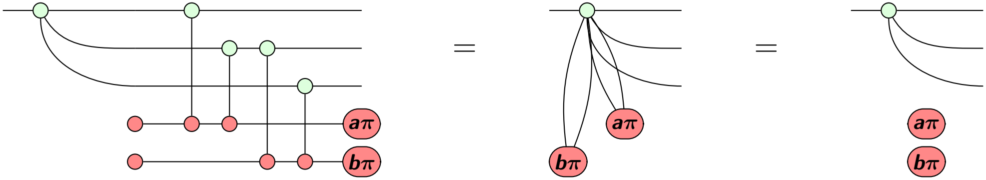


Encoding and decoding when there are no errors yields outcomes $a = 0 = b$:



The detectors

No errors:



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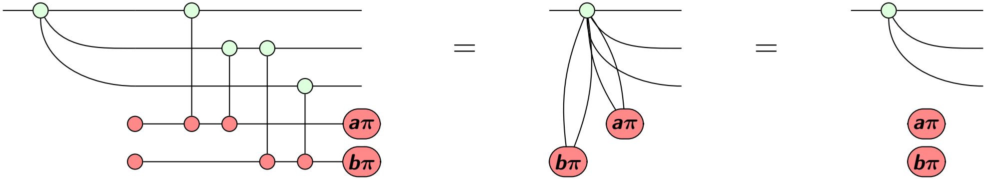
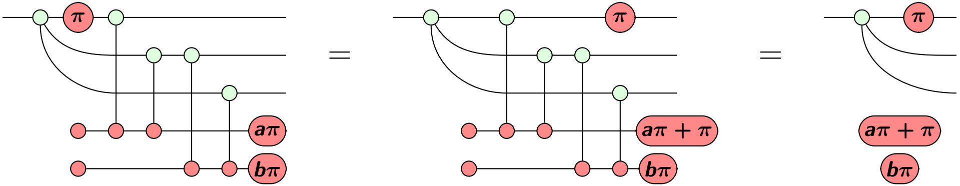


A bit-flip error on the first qubit changes the first parity measurement:



Fault-tolerant computation on encoded data

Transversality: each physical gate should only touch one of the encoded qubits to avoid spreading errors.

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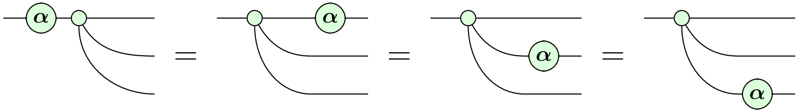

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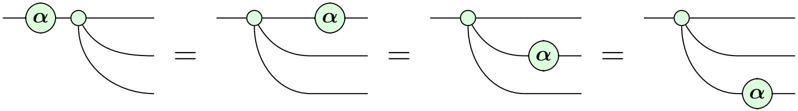
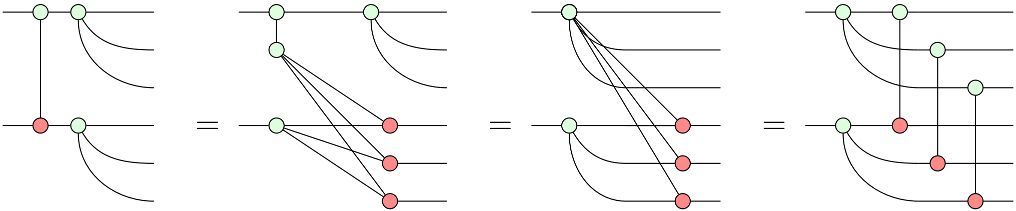


Logical controlled-X gate:



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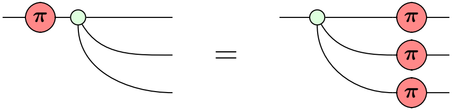

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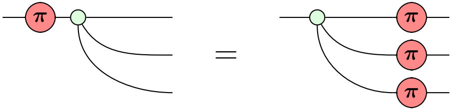
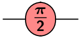
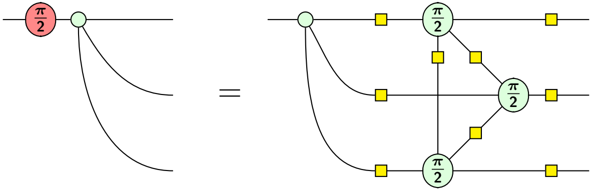


Other X-rotations are not transverse, e.g. for :

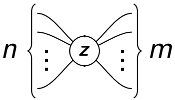


Need to use more complicated techniques to implement these gates fault-tolerantly.

Outline

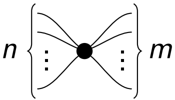
The ZW-calculus

Z spider, $z \in \mathbb{C}$ [Hadzihasanovic 2015; Hadzihasanovic, Ng, Wang 2018]

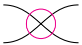


$$\rightsquigarrow \quad |\underbrace{0\ldots0}_{m}\rangle\langle\underbrace{0\ldots0}_{n}| + z|\underbrace{1\ldots1}_{m}\rangle\langle\underbrace{1\ldots1}_{n}| = \begin{pmatrix} 1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & z \end{pmatrix}$$

W pseudo-spider



Fermionic swap



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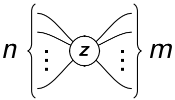

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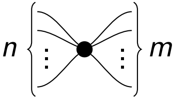

$$\rightsquigarrow \quad \sum |\mathbf{x}\rangle\langle\mathbf{y}|, \qquad \text{where } \mathbf{x} \in \{0,1\}^m, \mathbf{y} \in \{0,1\}^n, |\mathbf{xy}| = 1$$

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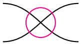

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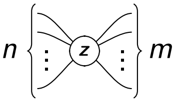

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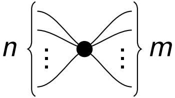

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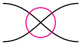
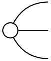
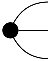


$$\rightsquigarrow \quad |00\rangle\langle00| + |01\rangle\langle10| + |10\rangle\langle01| - |11\rangle\langle11| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Some properties of the ZW-calculus

The two three-legged spiders of the ZW-calculus represent the two inequivalent types of 3-qubit entangled states:

 $\rightsquigarrow \quad |000\rangle + |111\rangle$

 $\rightsquigarrow \quad |001\rangle + |010\rangle + |100\rangle$

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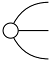
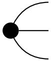
 $\rightsquigarrow \quad |000\rangle + |111\rangle$

 $\rightsquigarrow \quad |001\rangle + |010\rangle + |100\rangle$

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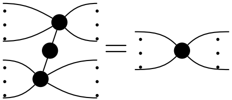

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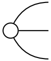
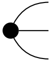
 $\rightsquigarrow \quad |000\rangle + |111\rangle$

 $\rightsquigarrow \quad |001\rangle + |010\rangle + |100\rangle$

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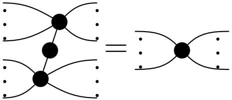
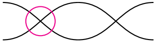
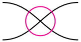


The fermionic swap is not ‘flexsymmetric’ – unlike for other generators, swapping its ‘legs’ changes the operation:

 $\neq$ 

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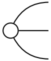

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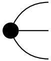

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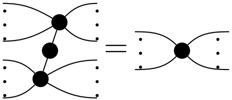

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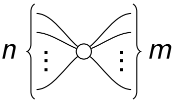
The ZW-calculus has a complete equational theory, in fact it was the first universal graphical calculus to be proved complete.

The ZH-calculus

Z spider

[B. & Kissinger 2018]



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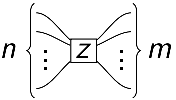
H box, where $z \in \mathbb{C}$



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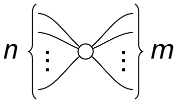

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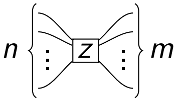

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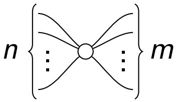

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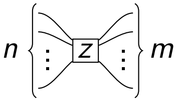

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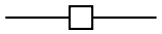
Some properties of the ZH-calculus

The binary H-box with label -1 is a Hadamard gate (up to scaling):



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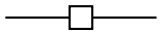

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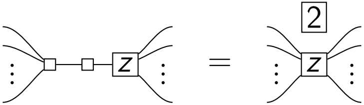

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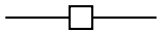

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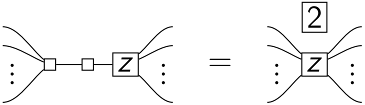
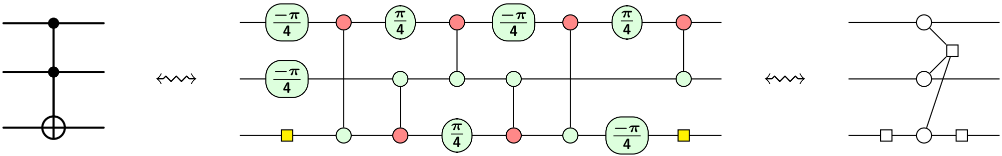


The ZH-calculus is particularly useful for representing 'classical non-linearity', such as in multi-controlled gates, for example the Toffoli gate:



Probabilistic mixtures of quantum states

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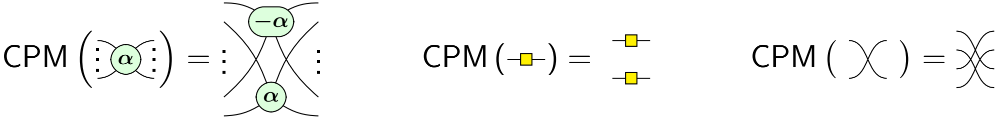
The above examples become:

$$|+\rangle\langle+| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Graphical calculi for mixed states and classical information

Change the interpretation of the existing generators by combining each generator with its complex conjugate, e.g. for the ZX-calculus:



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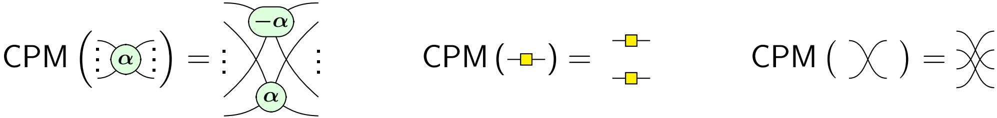
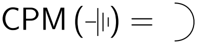


Add one new generator, which represents ‘measure and delete outcome’



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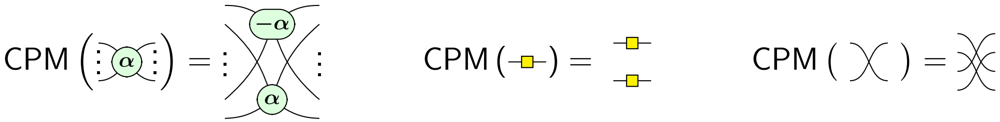

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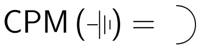
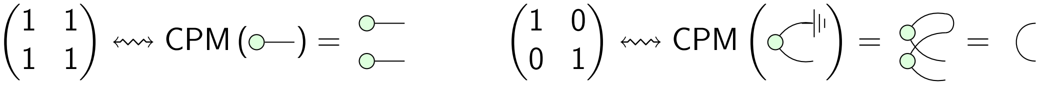
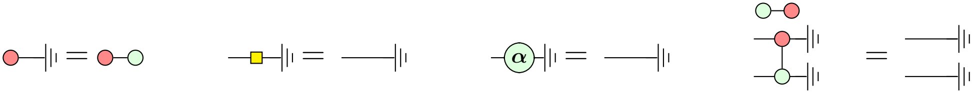


Then we can represent superpositions and probabilistic mixtures:



Equations for the new generator, and application

All existing ZX-calculus equations remain true and it suffices to add four new equations involving  [Carette et al. 2019]



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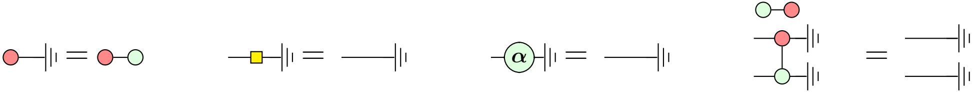
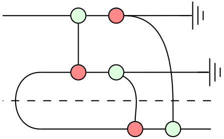


For example, this allows to represent the classical information flow in quantum teleportation:



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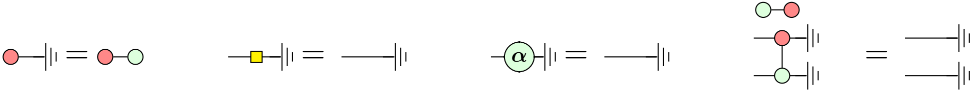

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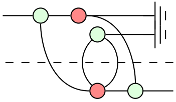

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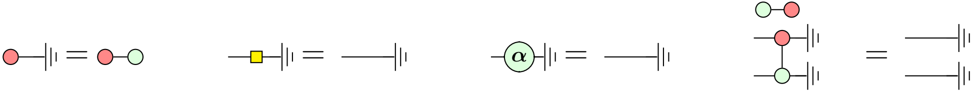

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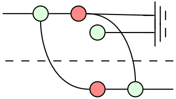

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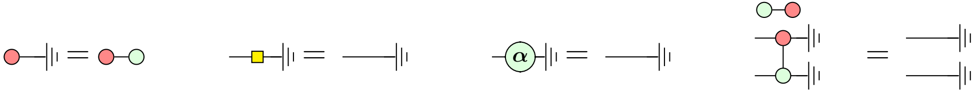

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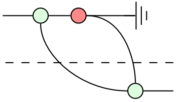

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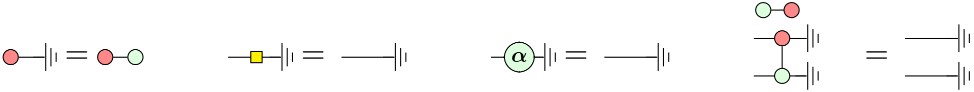

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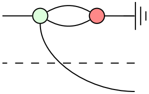

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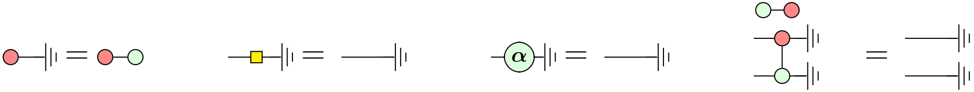

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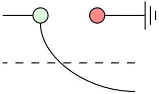

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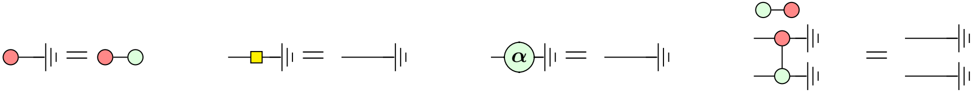

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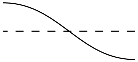

Scalable notation

Bundle multiple wires into one and introduce two new components to gather and split these bundles:

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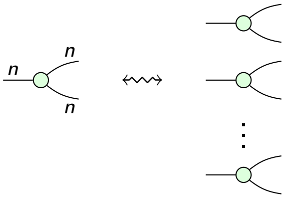
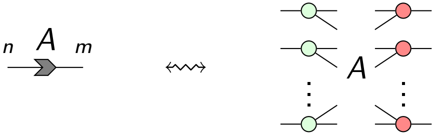
In addition to combining multiple wires, also allow multiple copies of other generators, e.g.



[Carette, Horsman, Perdrix 2019]

Unified notation and reasoning for diagrams with similar structure



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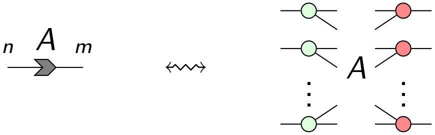

Applying the matrix arrow labelled by a biadjacency matrix A to a computational basis state $|\mathbf{x}\rangle$ for some $\mathbf{x} \in \{0,1\}^n$:

$$|\mathbf{x}\rangle \xrightarrow{\quad A \quad} |A\mathbf{x}\rangle$$

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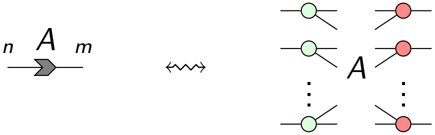

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This has applications in quantum error correcting codes and many other areas of quantum computing.

Outline

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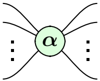

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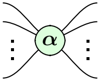

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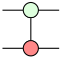

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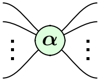

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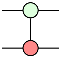

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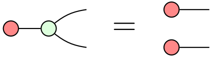

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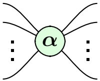

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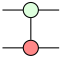

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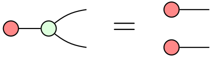

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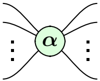

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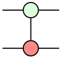

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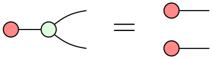

More information about the ZX-calculus is at `https://zxcalculus.com/`. See also ⟨hal-05322779⟩ (in French).